Novel Meshes for Multivariate Interpolation and Approximation

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Motivation

Regression and interpolation are problems of considerable importance that find applications across many fields of science.

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These techniques are applied here to:

- High performance computing file input/output (HPC I/O)
- Parkinson's patient clinical evaluations
- Forest fire risk assessment

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Problem Formulation

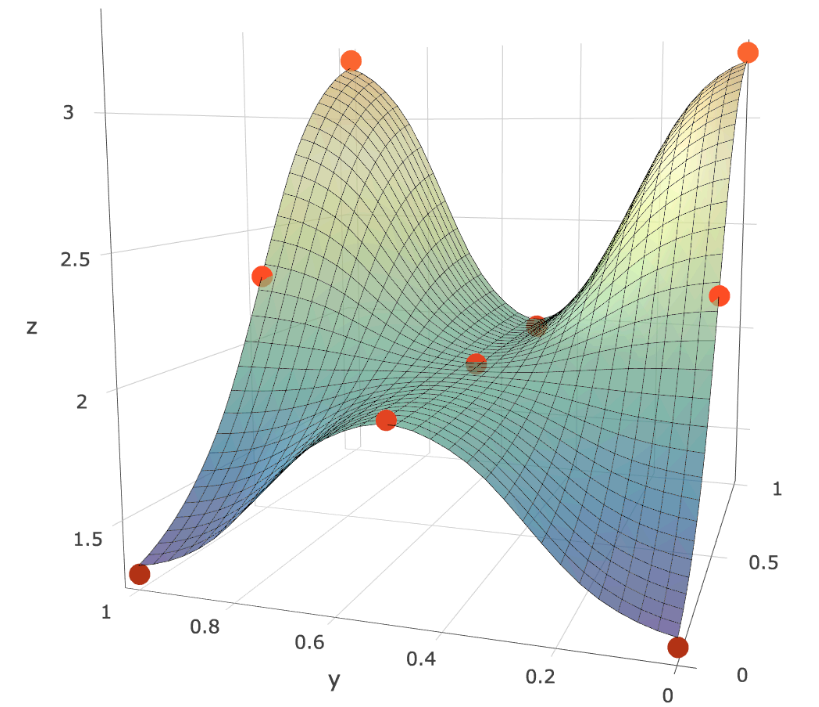
Given

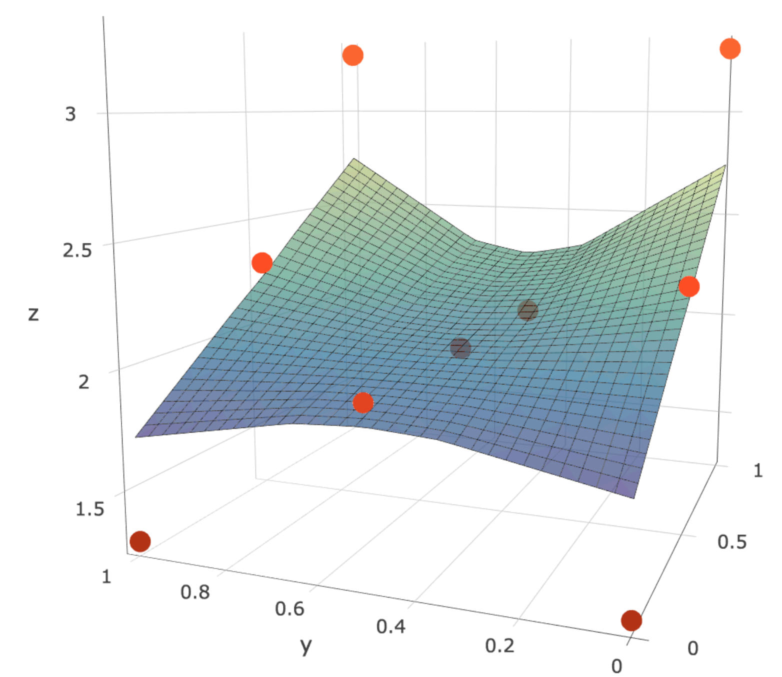
underlying function $f : \mathbb{R}^{d} \rightarrow \mathbb{R}$

data matrix $X^{n \times d}$ with row vectors $x^{(i)} \in \mathbb{R}^{d}$

response values $f(x^{(i)})$ for all $x^{(i)}$

matrix $f(X)$ has rows $f(x^{(i)})$





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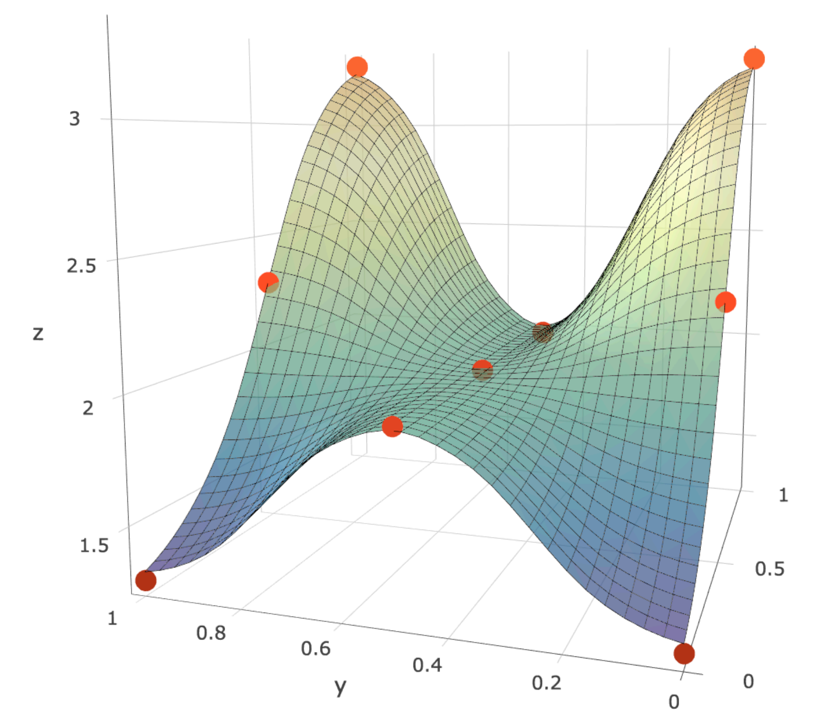

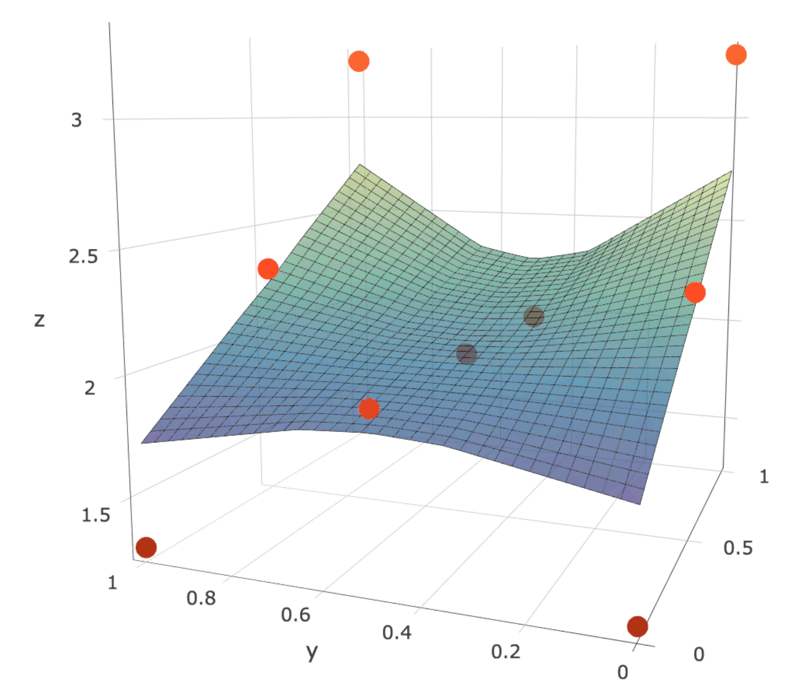

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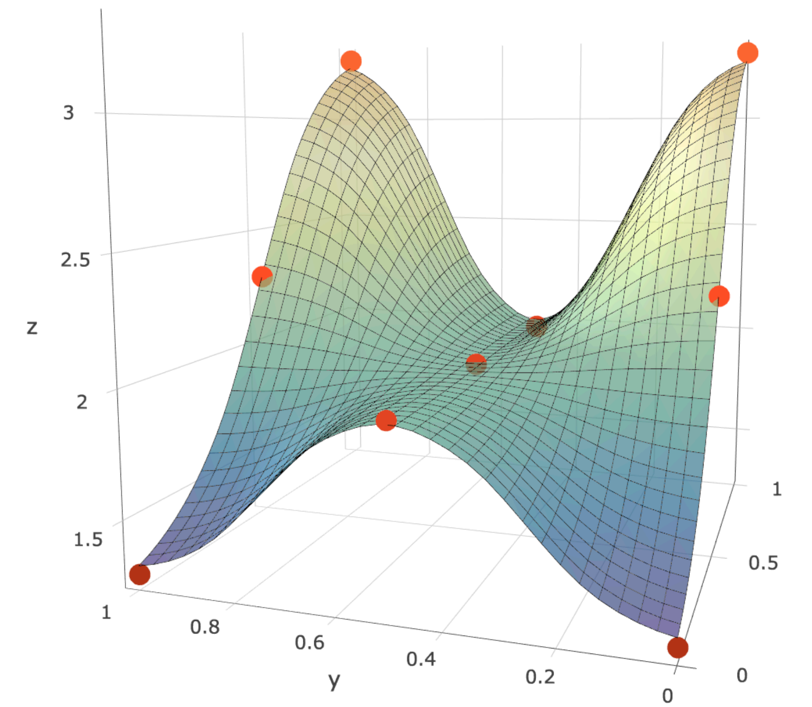

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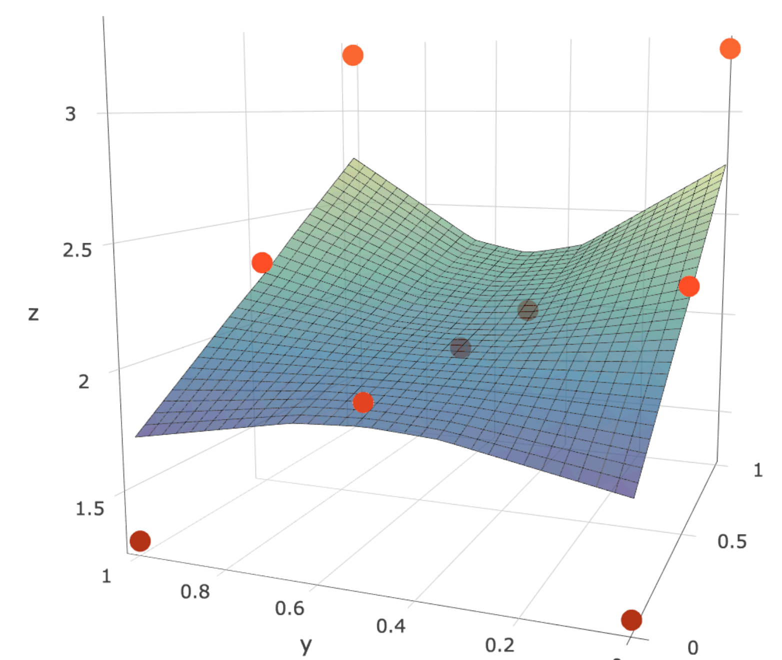

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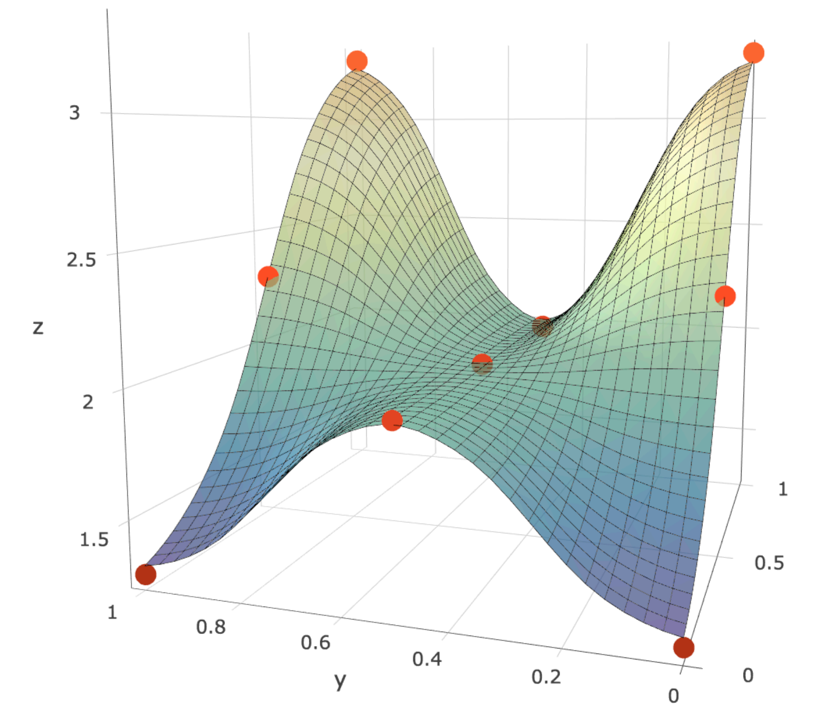

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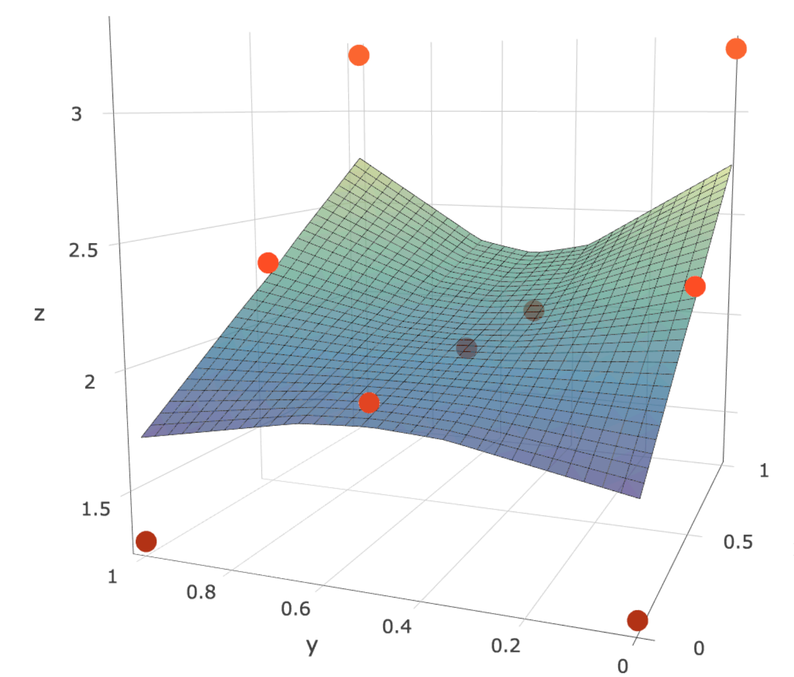
Approximation

g has parameters P and is the solution to $\min_P \| f(X) - g(X) \|$



Box Splines

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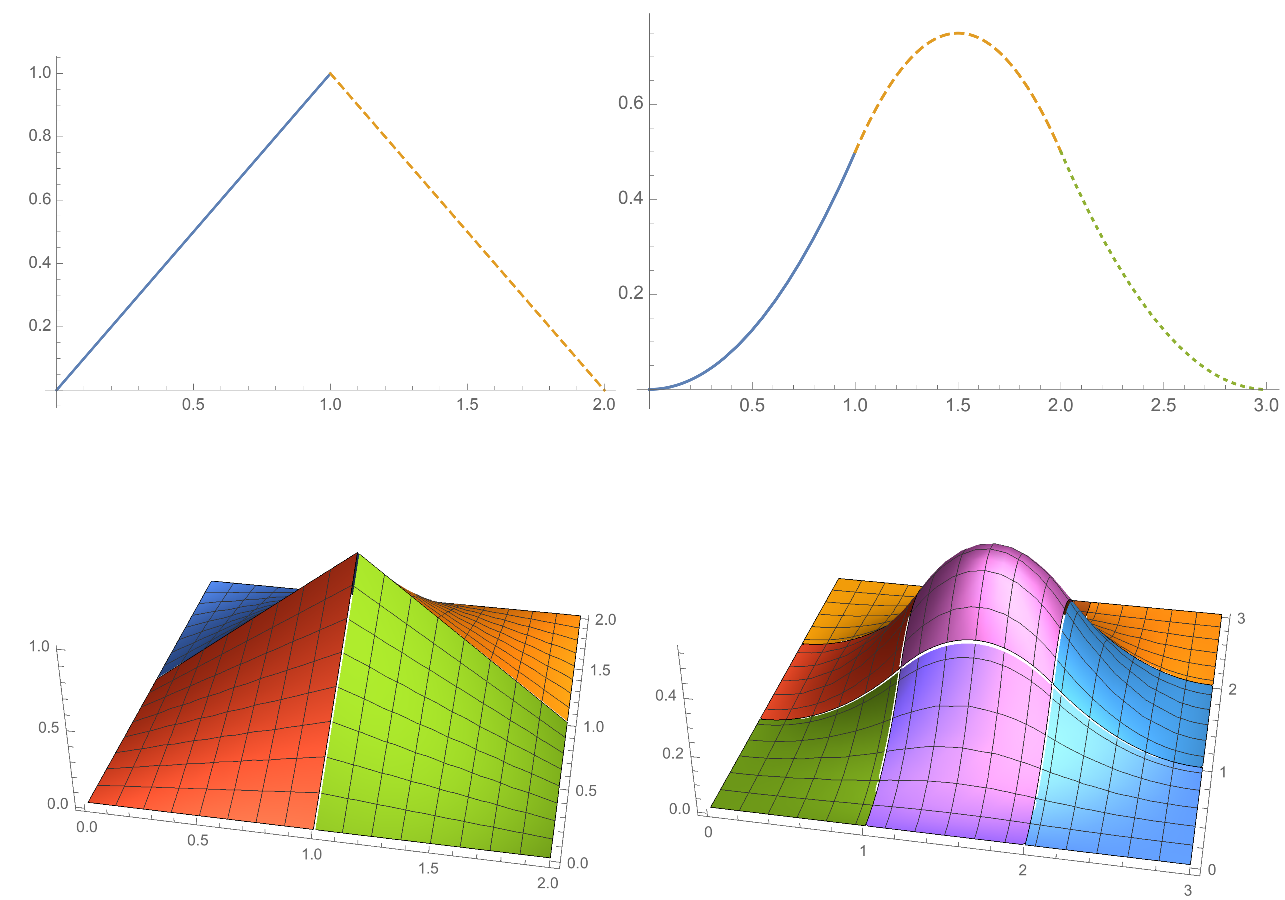

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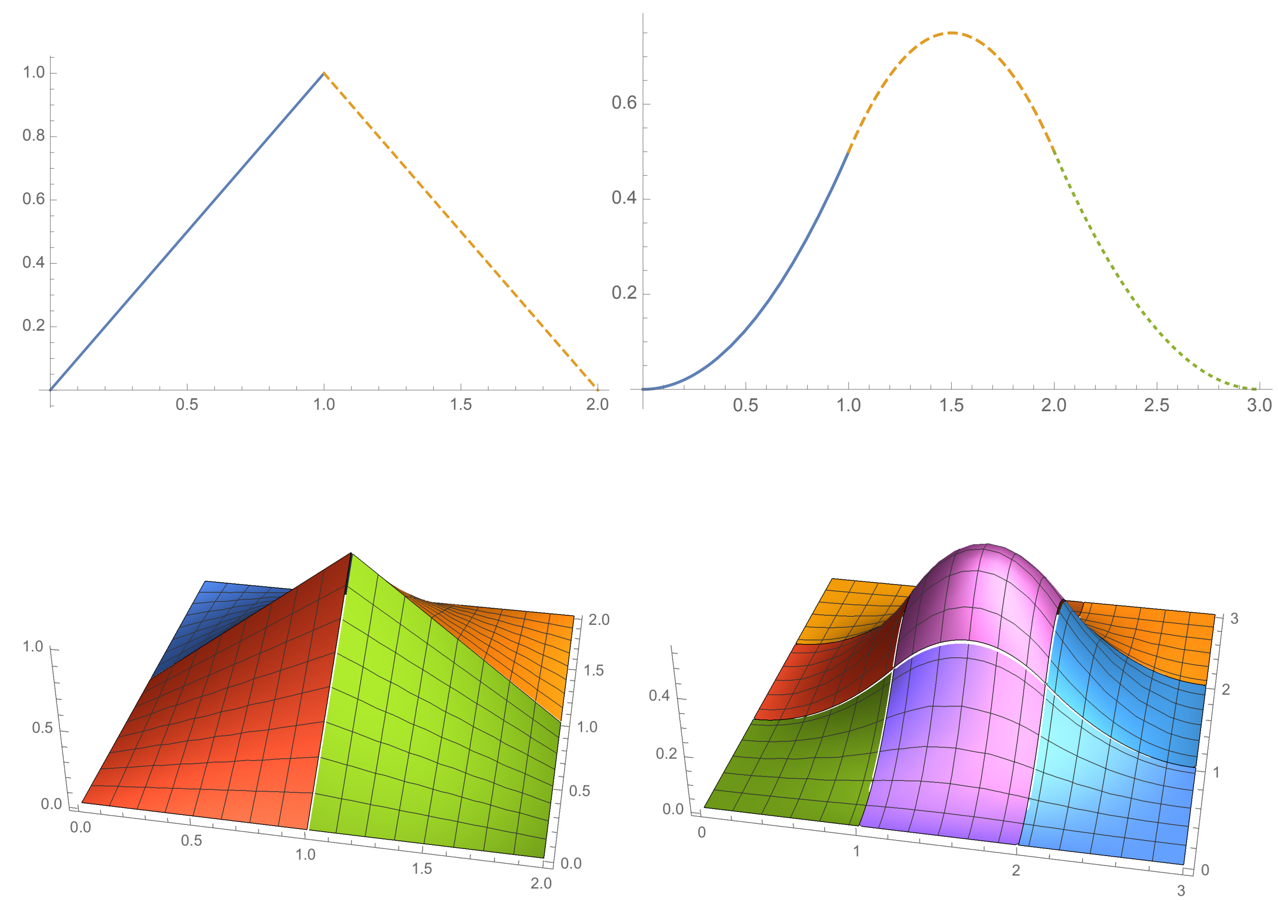

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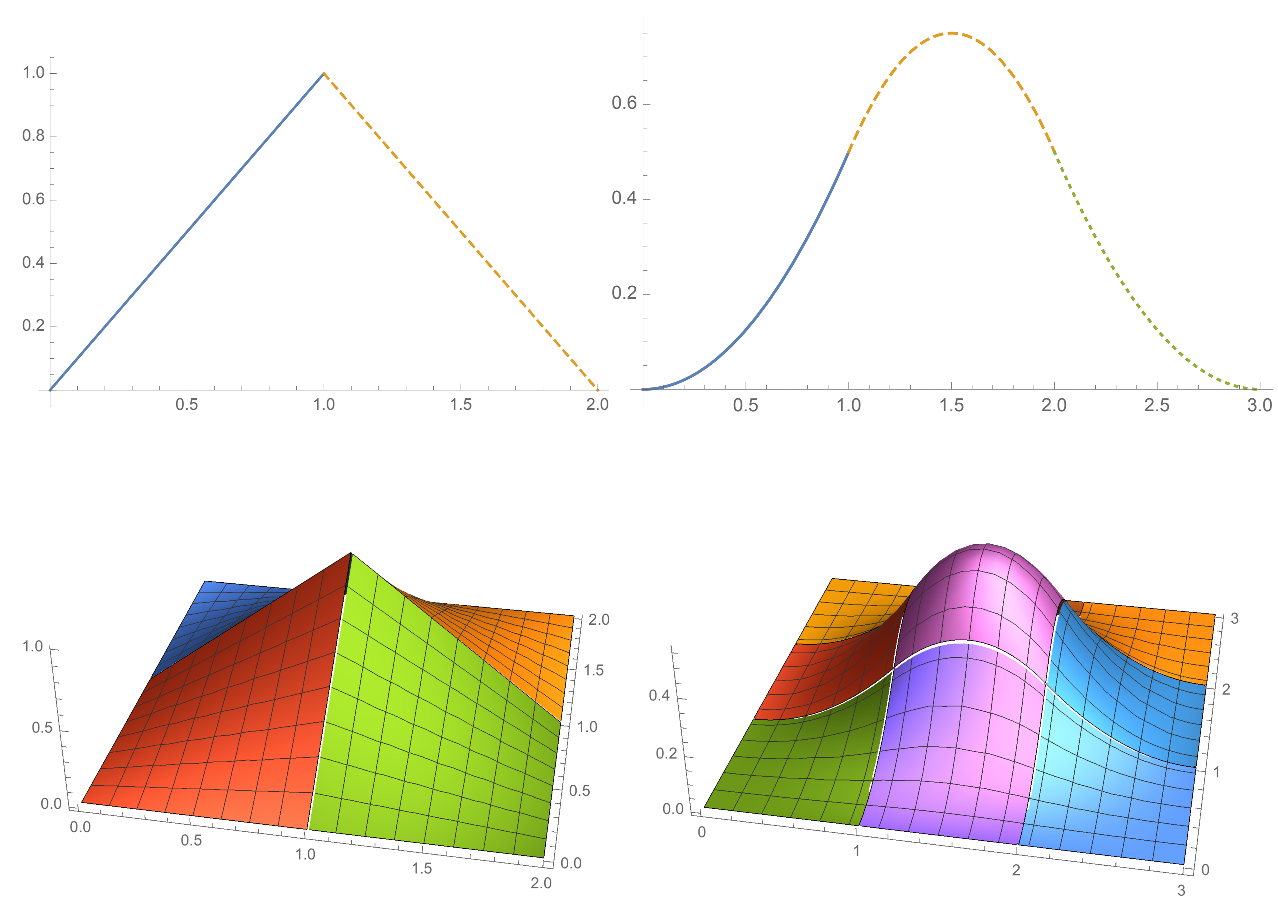

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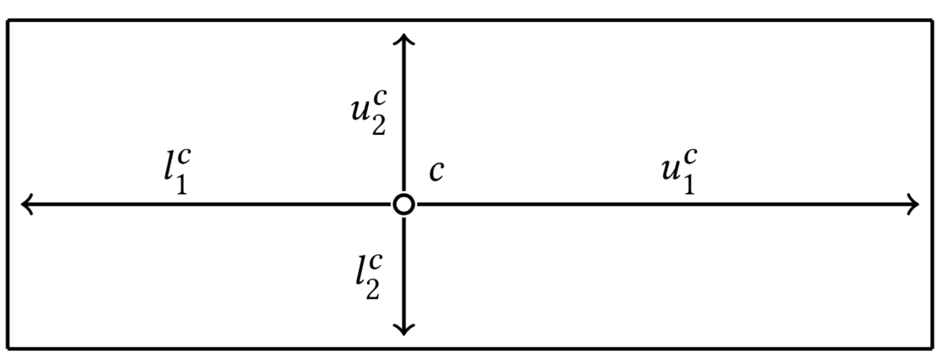
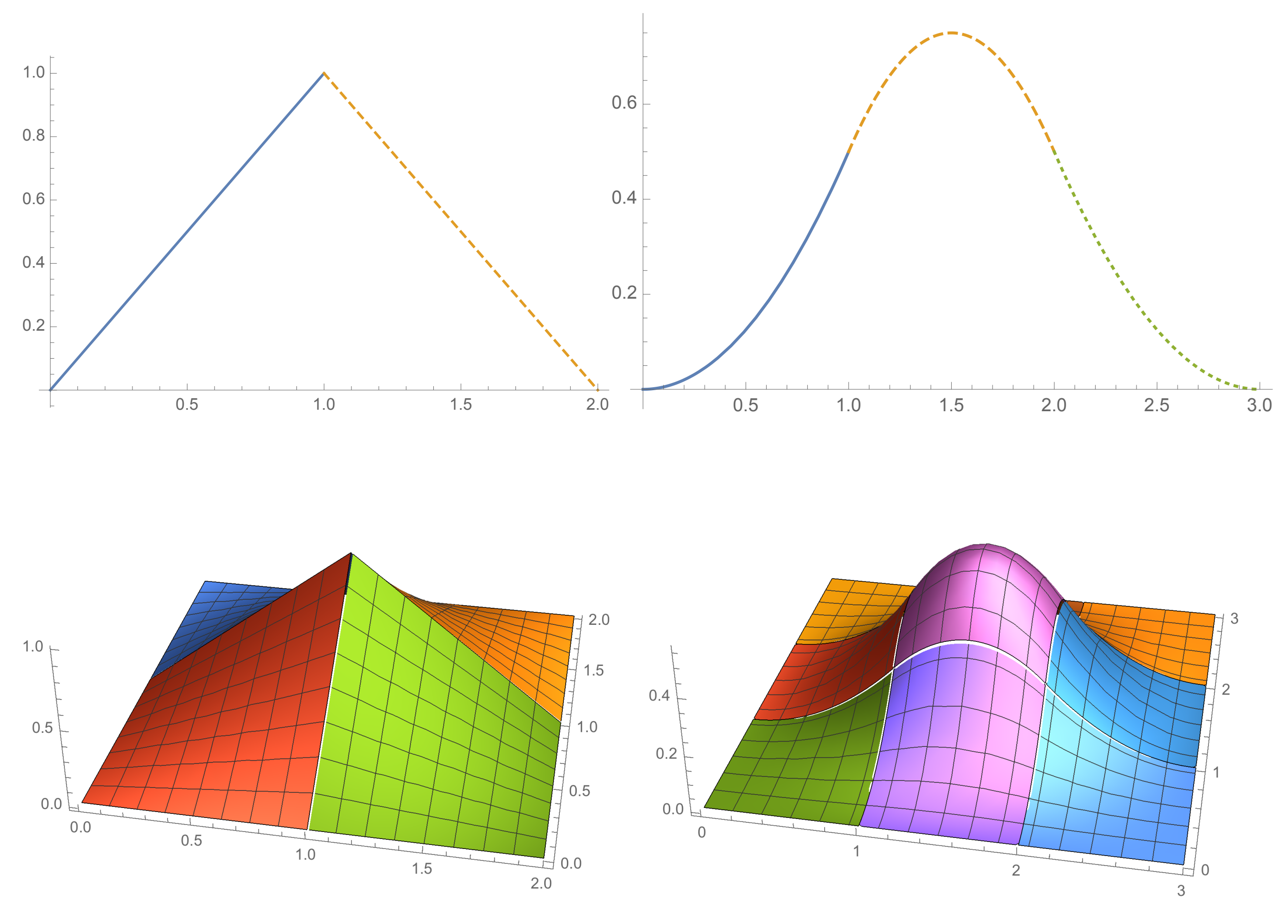




Max Box Mesh (MBM)

Properties

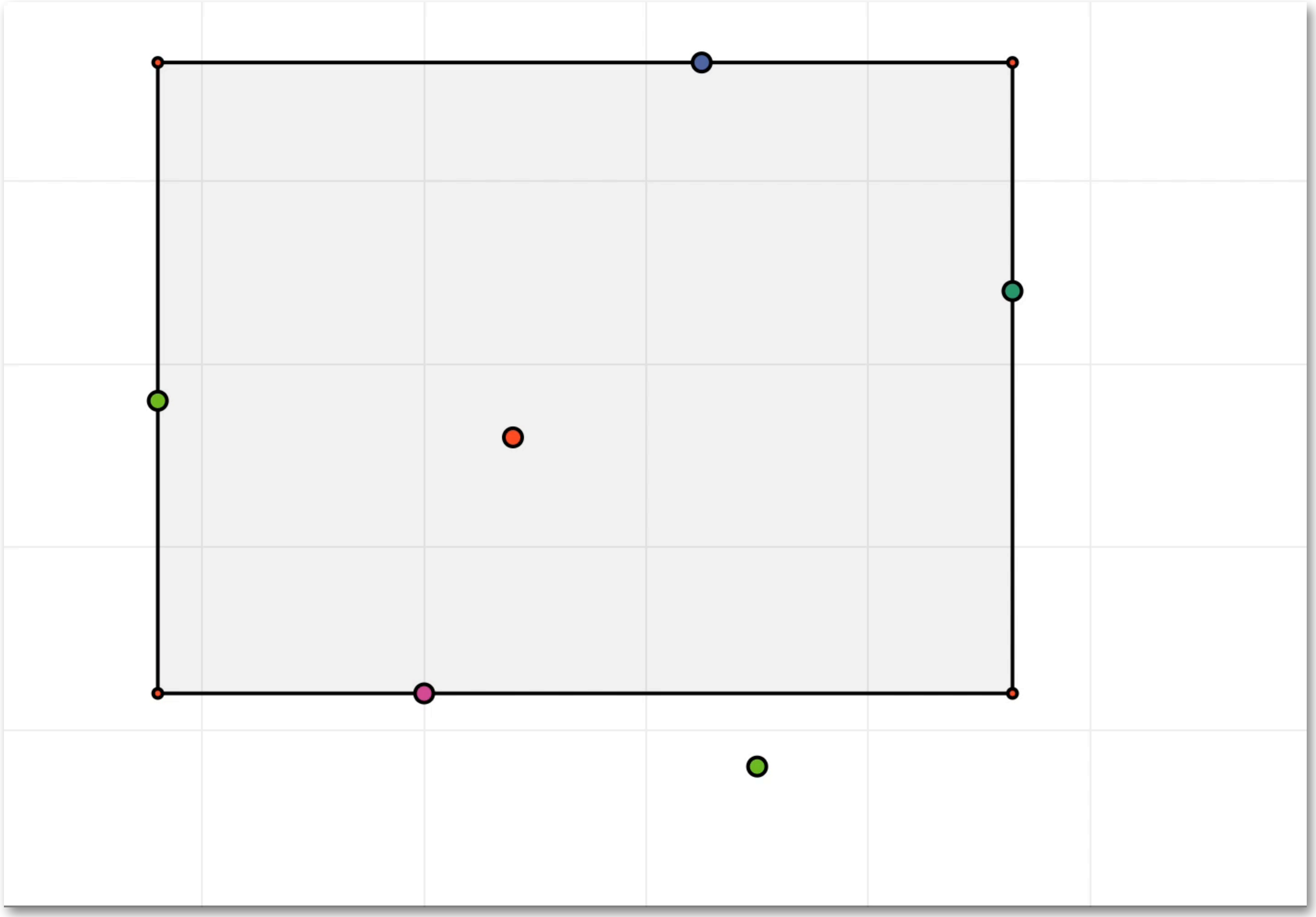
Largest max norm distance from anchor to edge of support.

Not always a covering for the space.

Construction Complexity

For each box (n)
  Distance to all (nd)
  For each dimension (2d)
    Sort distances (n log n)

$\mathcal{O}(n^2 d \log n)$



Iterative Box Mesh (IBM)

Properties

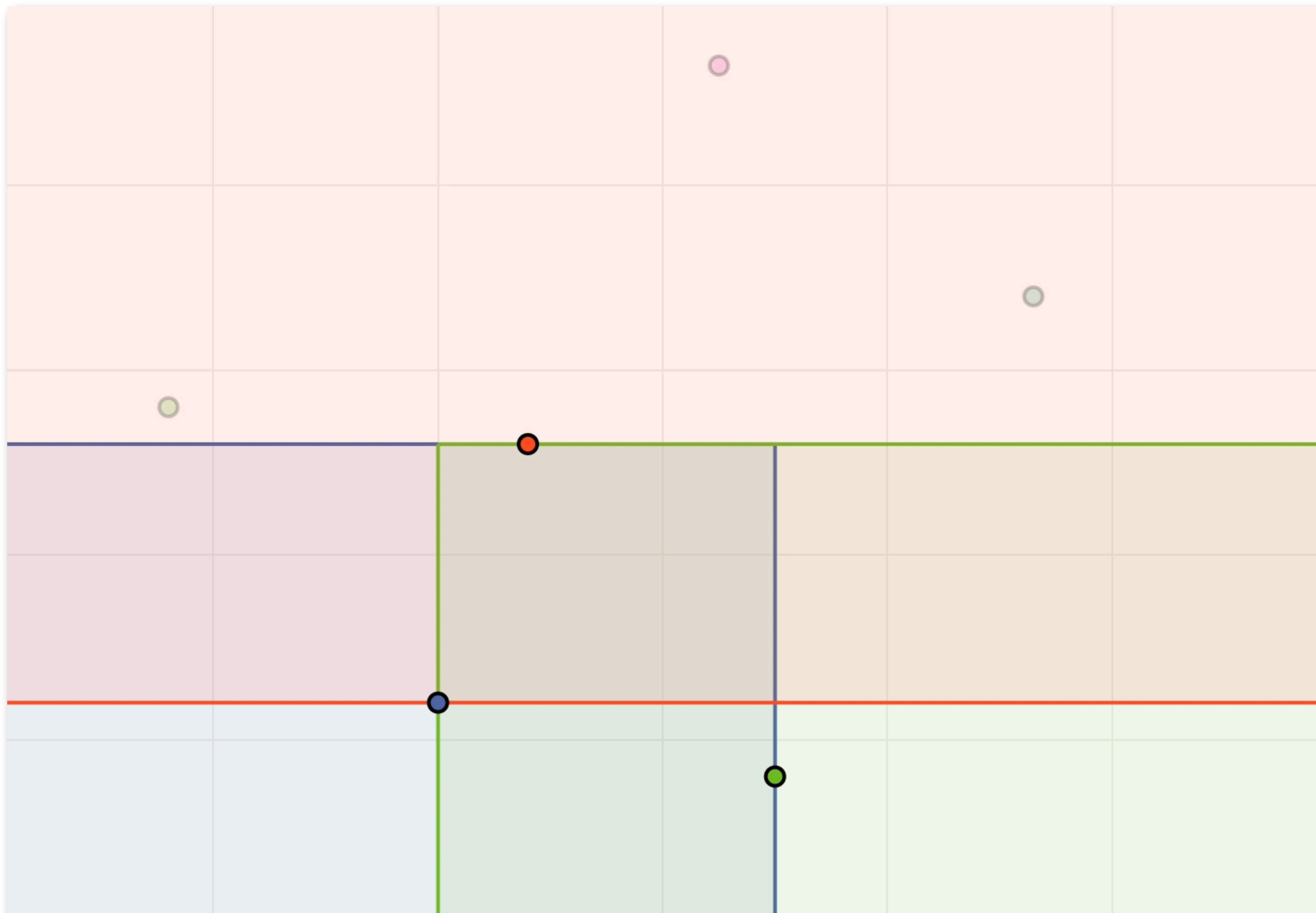
Built-in bootstrapping used to guide construction.

Always a covering for the space by construction.

Construction Complexity

Until all points are added (n)
Identify boxes containing new anchor to add (n)
Shrink boxes containing new anchor along all dimensions (d)

$$\mathcal{O}(n^2 + nd)$$



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Voronoi Mesh (VM)

Properties

Naturally shaped geometric regions (not forcibly axis aligned)

Always a covering for the space by construction.

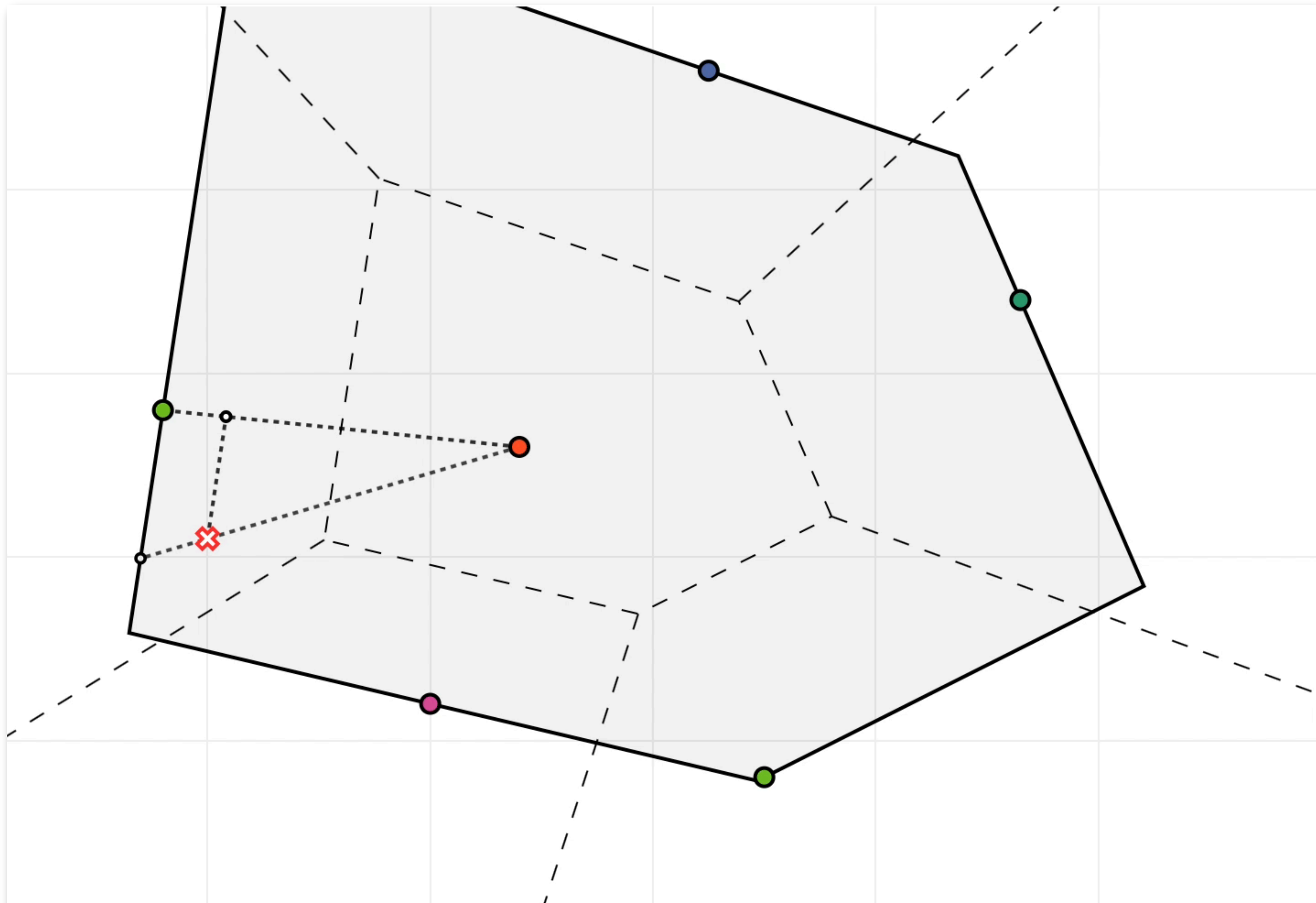
Construction Complexity

$$\mathcal{O}(n^2 d)$$

Prediction Complexity

For each cell anchor (n)
  For each other anchor, compute distance (nd)

$$\mathcal{O}(n^2 d)$$



Fitting and Bootstrapping

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Evaluate all basis functions in the mesh at all points *n*.

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When “c” is the number of control points used for a mesh, using an (n × c) matrix *A* of basis function evaluations at all points, solve the least squares problem $A\,x = f\,(\mathrm{X})$ with cost $\mathcal{O}(nc^2 + c^3)$.

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Initialize mesh only using the most central point

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Increased cost up to $\mathcal{O}(\mathrm{n})$

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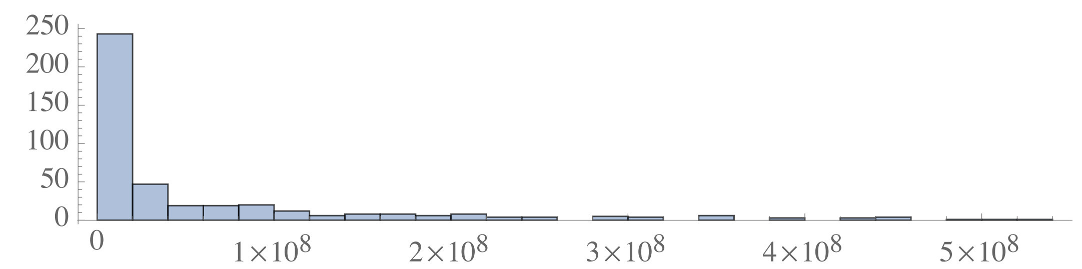
High Performance Computing File I/O

$n = 532, d = 4$

predicting *file I/O throughput*



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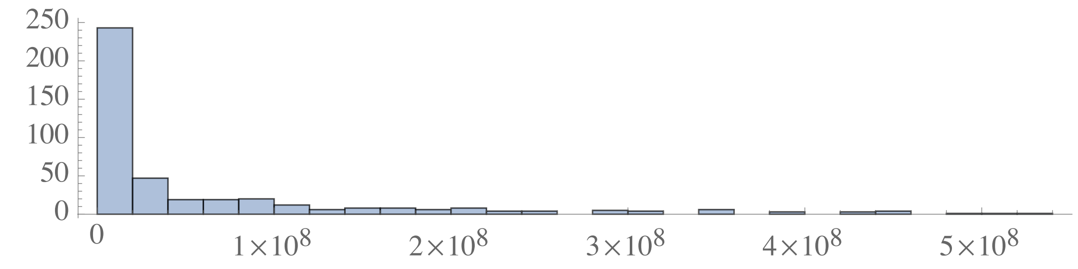
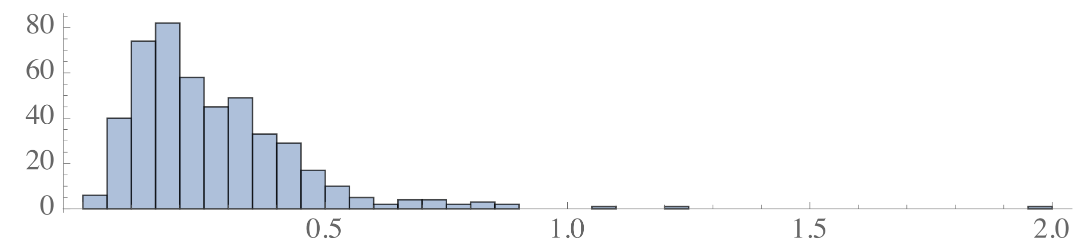


Forest Fire
$n = 517, d = 12$
predicting *area burned*



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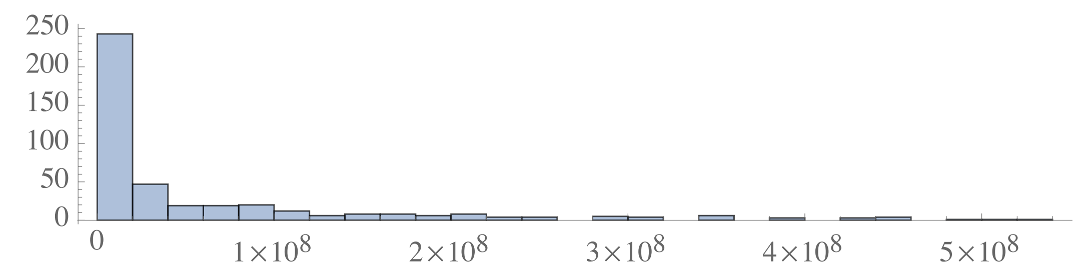

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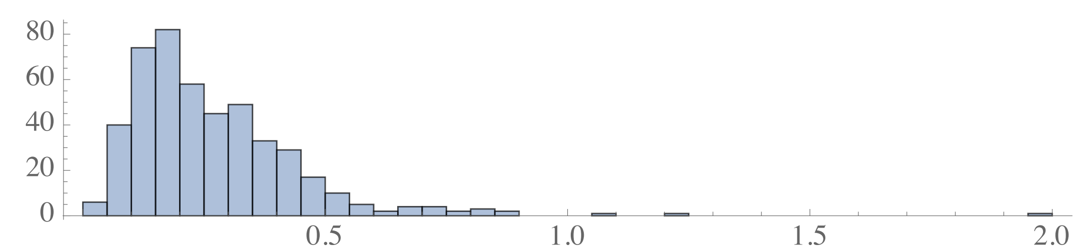
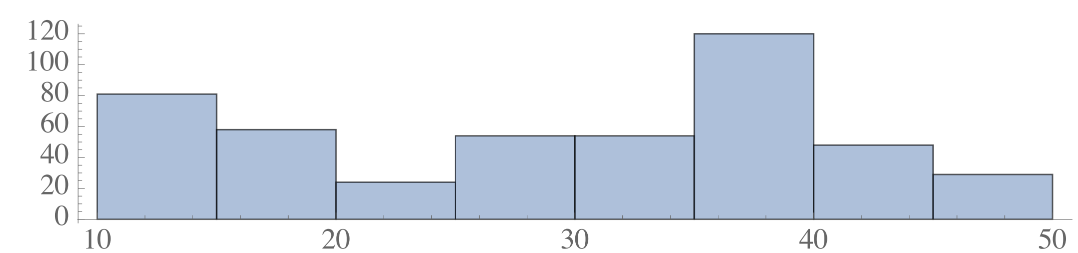


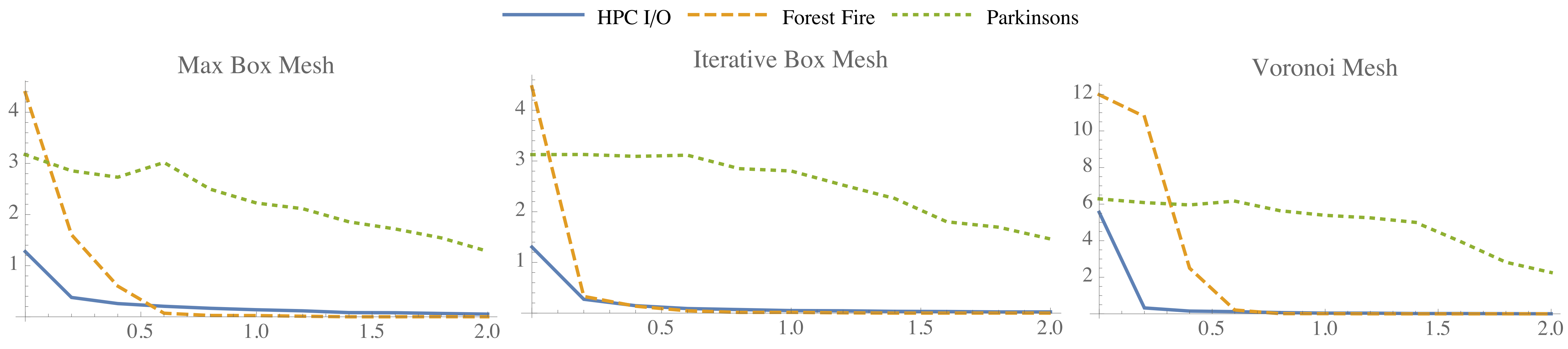
Parkinson's Clinical Evaluation

$n = 468, d = 16$

predicting *total clinical "UPDRS" score*



Time to Fit (y-axis) versus Error Tolerance (x-axis)



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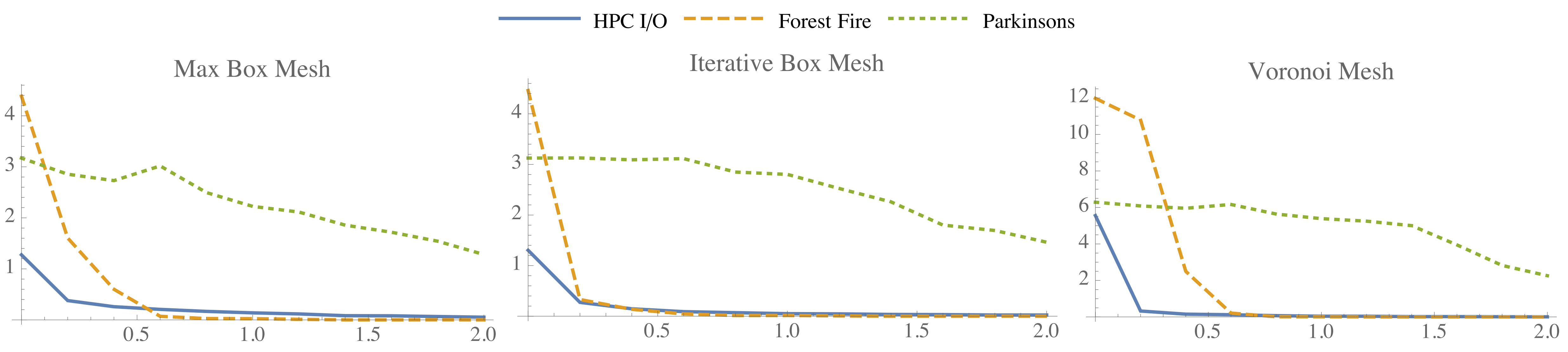
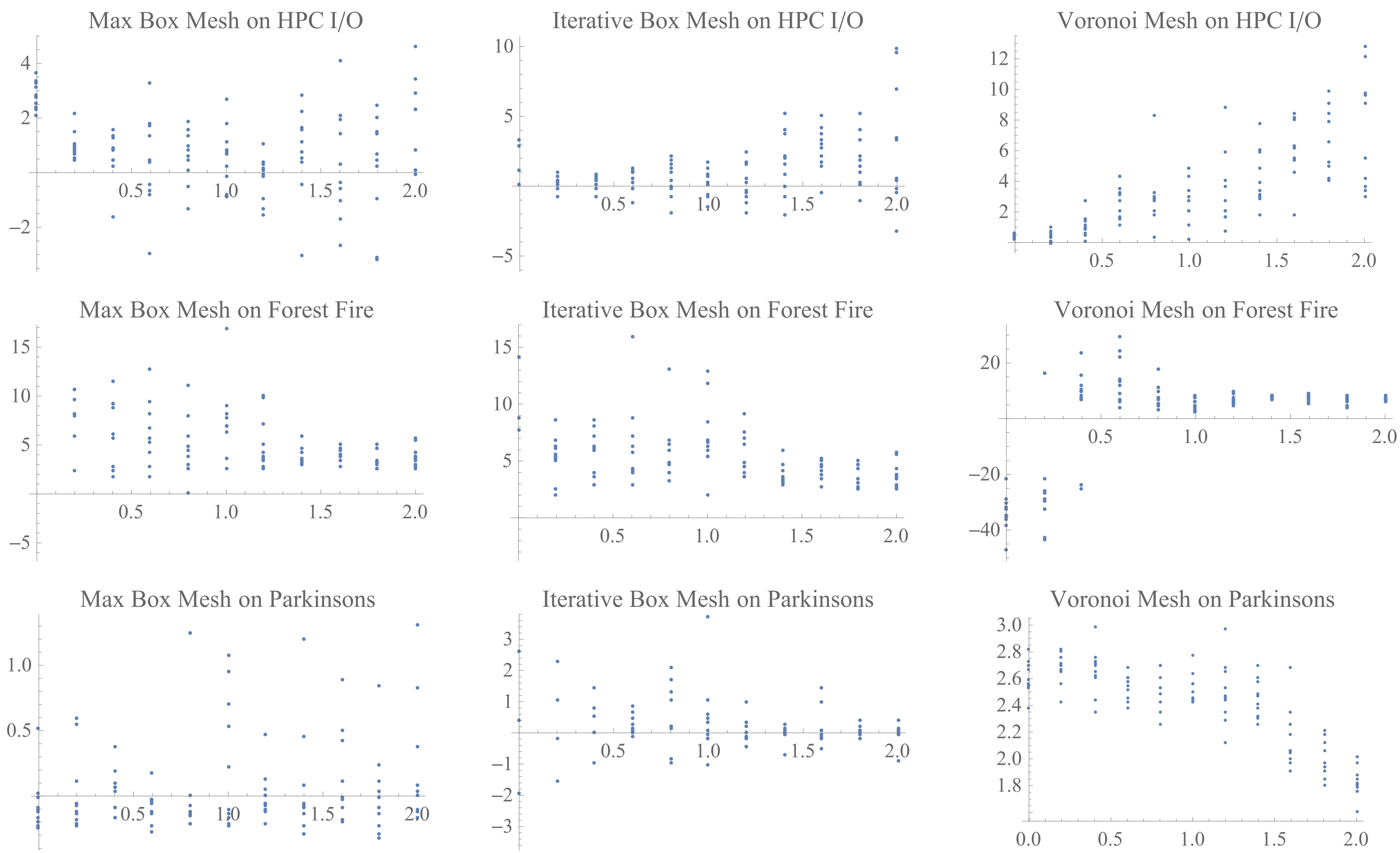


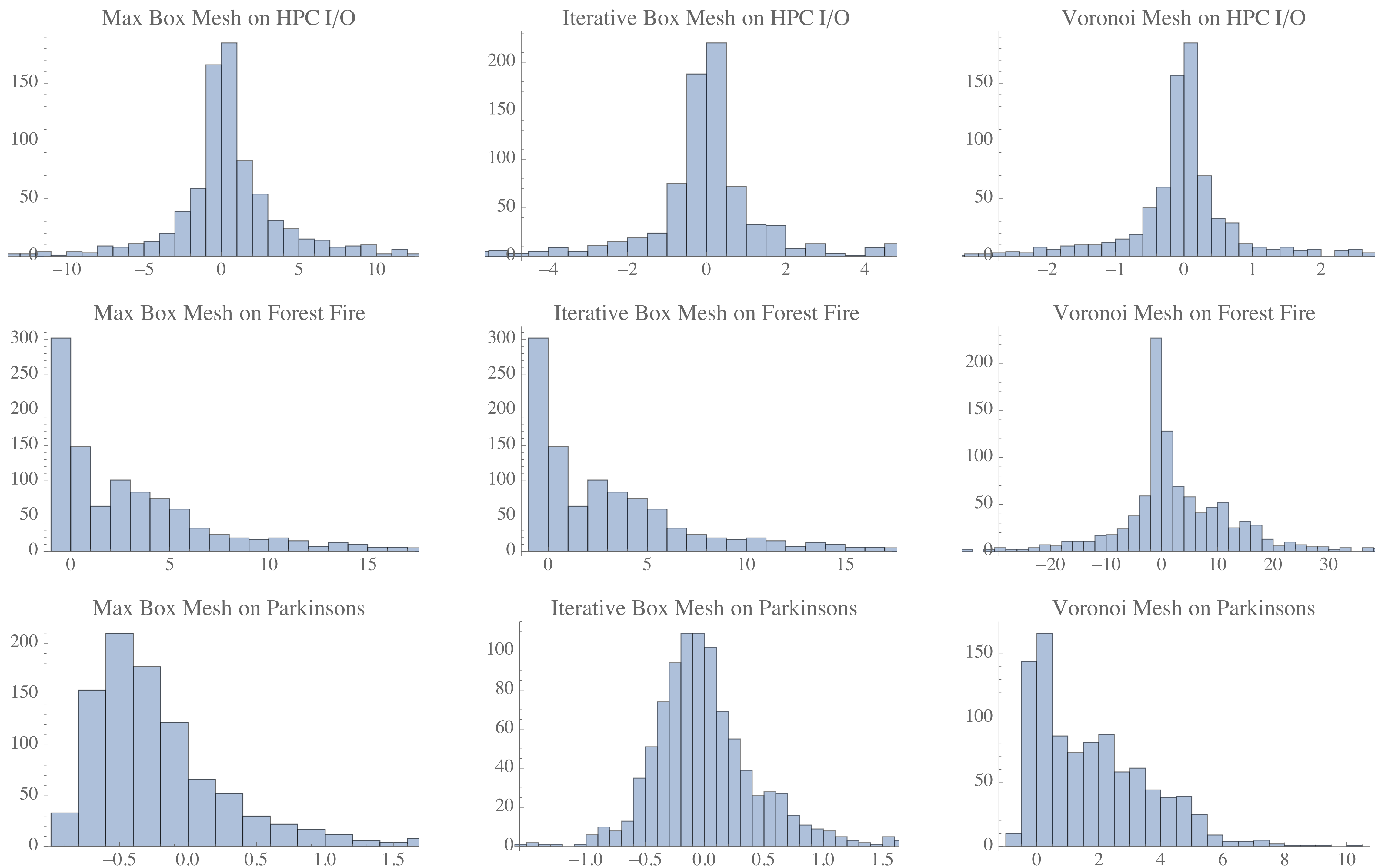
Optimal Tolerance & Accuracy

Data Set	Technique	Tolerance	Average Error
HPC I/O	MBM	1.2	0.597
Forest Fire	MBM	1.8	3.517
Parkinson’s	MBM	0.6	0.114
HPC I/O	IBM	0.4	0.419
Forest Fire	IBM	1.8	3.615
Parkinson’s	IBM	1.8	0.121
HPC I/O	VM	0.2	0.382
Forest Fire	VM	1.0	4.783
Parkinson’s	VM	2.0	1.824

Average Relative Testing Error (y-axis) versus Relative Error Tolerance (x-axis)



Histograms of Signed Relative Error



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Each is *theoretically straightforward*, *flexible*, and suitable for applications in *many dimensions*.

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Future Work

Alternative smoothing methods may be preferred that scale better with the number of data points.

Further theoretical and empirical comparisons may be done against more widely used statistical / ML techniques.